Translations of Linear Functions

ABOUT THIS LESSON
This lesson develops an appreciation for the power and efficiency of the point-slope form of a linear equation. By examining the effects of horizontal and vertical translations of linear functions, and by factoring with a common constant factor, students make the connection between the slope-intercept and point-slope forms and can begin to see the point-slope form as a translation of the linear parent function. They also experience the ease of using the point-slope form both for graphing and for writing equations.

OBJECTIVES
Students will
- graph linear equations.
- translate graphs of linear functions horizontally and vertically.
- factor a common integer factor from a simple expression.
- identify a point and a slope from an equation in point-slope form.

LEVEL
Grade 8 or Algebra 1 in a unit on writing linear equations

MODULE/CONNECTION TO AP*
Analysis of Functions: Transformations
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MODALITY
NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical
COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (*) at the end of a specific standard indicates that the high school standard is connected to modeling.

Targeted Standards (if used in Grade 8)

8.G.1a: Verify experimentally the properties of rotations, reflections, and translations: (a) Lines are taken to lines, and line segments to line segments of the same length.
See questions 1-16

8.F.2: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).
For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.
See questions 3-16

8.F.4: Construct a function to model a linear relationship between two quantities.
Determine the rate of change and initial value of the function from a description of a relationship or from two \((x, y)\) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
See questions 1c, 2c, 3d, 11, 12, 14

Reinforced/Applied Standards (if used in Grade 8)

8.F.3: Interpret the equation \(y = mx + b\) as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.
For example, the function \(A = s^2\) giving the area of a square as a function of its side length is not linear because its graph contains the points \((1,1), (2,4)\) and \((3,9)\), which are not on a straight line.
See questions 1a, 2a, 4c, 5c, 6a, 7, 8, 9a, 14, 15a-b

7.EE.2: Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. For example, \(a + 0.05a = 1.05a\) means that “increase by 5%” is the same as “multiply by 1.05.”
See questions 1f, 2f
Targeted Standards (if used in Algebra 1)

F-BF.3: Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( k f(x) \), \( f(kx) \), and \( f(x+k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

See questions 1b, 1e, 2b, 2e, 3b, 4a-b, 5a-b, 8b-c, 9b-d, 10b-c, 11-12, 16

A-CED.2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.*

See questions 1a, 1c, 2a, 2c, 4c, 5c, 6a, 7, 8a, 9a, 15a-b

A-SSE.2: Use the structure of an expression to identify ways to rewrite it. For example, see \( x^4 - y^4 \) as \( (x^2)^2 - (y^2)^2 \), thus recognizing it as a difference of squares that can be factored as \( (x^2 - y^2)(x^2 + y^2) \).

See questions 1d, 1f, 2d, 2f, 3a, 3c, 4d, 5d, 10c, 13-14

Reinforced/Applied Standards (if used in Algebra 1)

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximaums and minimums; symmetries; end behavior; and periodicity.*

See questions 1b, 2b, 7d, 8b, 9b, 10a

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.7: Look for and make use of structure.

Students recognize the relationship between applying translations to parent functions and graphing using point-slope form. They use this relationship to quickly and accurately graph linear functions rather than converting to the commonly used slope-intercept form.

MP.8: Look for and express regularity in repeated reasoning.

After examining multiple translations, students recognize a pattern that enables them to graph and write equations quickly using point-slope form of a line.
FOUNDATIONAL SKILLS
The following skills lay the foundation for concepts included in this lesson:

- Determine points based on an equation
- Graph a linear function

ASSESSMENTS
The following types of formative assessments are embedded in this lesson:

- Students engage in independent practice.
- Students summarize a process.

The following additional assessments are located on our website:

- Analysis of Functions: Transformations – Algebra 1 Free Response Questions
- Analysis of Functions: Transformations – Algebra 1 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Colored pencils
- Interactive applet which demonstrates transformational changes in functions of the form $y = d \cdot f \left( c \left( x - a \right) \right) + b$:
  
  http://www.sfu.ca/~jtmulhol/calculus-applets/
  GeoGebra-Worksheets/graphing-transformations.html
TEACHING SUGGESTIONS

This is a discovery activity that can be effectively done in groups. Remind students as they begin question 1 to use ordered pairs which include both positive and negative x values as they graph the lines. As the students progress through the activity, check that they understand the concepts and encourage them to discuss the results of each question. The activity begins by horizontally translating linear functions and introducing the effect that the translation has on the equation. The lesson continues to translate both vertically and horizontally and to examine the effects on the equation.

Through the translated equations, students are also introduced to the point-slope form of the equation of a line. The equations are written in the forms

\[ y = m(x - h) + k \]

and

\[ y = m(x + h) + k \]

instead of

\[ y = m(x - x_1) + y_1 \] or \[ y - y_1 = m(x - x_1) \]

in order to avoid the use of subscripts. Students should compare and contrast point-slope and slope-intercept form and understand the connection between the two forms. It is important for students to be introduced to and use point-slope form, because it translates most easily to the transformational form for other types of functions. The point-slope form is used in calculus to write the equation of a line tangent to a given curve at a given point. This form also leads to similar equations used in upper level math courses to graph more complicated equations based on parent functions such as \[ y = a(x - h)^2 + k \] where \((h, k)\) represents the vertex of the graph.

Students are introduced to \[ y = m(x - h) + k \], where \(h < 0\), to enable them to generalize the form with the subtraction sign. The activity can and should be completed without the use of graphing calculators for maximum conceptual understanding as students plot points and translate equations “by hand.”

Suggested modifications for additional scaffolding include the following:

1. Work question 1 as an example.
3, 7d,  Provide a fill-in-the-blank template.
10b
**NMSI CONTENT PROGRESSION CHART**

In the spirit of NMSI’s goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

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<thead>
<tr>
<th>6th Grade Skills/Objectives</th>
<th>7th Grade Skills/Objectives</th>
<th>Algebra 1 Skills/Objectives</th>
<th>Geometry Skills/Objectives</th>
<th>Algebra 2 Skills/Objectives</th>
<th>Pre-Calculus Skills/Objectives</th>
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<tbody>
<tr>
<td>Apply transformations to tessellations as well as to points, segments, and figures on the coordinate plane.</td>
<td>Apply transformations to tessellations as well as to points, segments, and figures on the coordinate plane.</td>
<td>Apply transformations including ( af(x - c) + d ) to linear, quadratic, exponential, piecewise, and generic functions.</td>
<td>Apply transformations to circles and apply transformations including ( af(x - c) + d ) to linear, quadratic, exponential, piecewise, and generic functions.</td>
<td>Apply transformations to conic sections and apply transformations including ( af(x - c) + d ) and compositions with absolute value including ( f(</td>
<td>x</td>
</tr>
</tbody>
</table>
Translations of Linear Functions

Answers

1. a. and b.
   c. \( y = 2x - 8 \)
   d. \( y = 2(x - 4) \)
   e. \((0, 0) \rightarrow (4, 0)\)
   
f. The amount of the translation appears inside the parentheses and after the subtraction sign.

2. a. and b.
   c. \( y = 2x + 8 \)
   d. \( y = 2(x + 4) \)
   e. \((0, 0) \rightarrow (-4, 0)\)
   
f. The amount of the translation appears inside the parentheses and after the addition sign.

3. a. One equation contains a subtraction sign inside the parentheses and the other one contains an addition sign.
   b. The equation with the addition sign translates the line \( y = 2x \) to the left, and the one with the negative sign translates the line to the right.
   c. Subtraction of a negative number is the same as addition of a positive number; therefore, the equations are the same, and the graphs will be the same.
   d. The common ratio is the slope of the lines.
4. c.
   a. The line is moved horizontally 2 units.
   b. The line is translated to the right.
   d. The $x$-coordinate is 2.

5. c.
   a. The graph is moved horizontally 2 units.
   b. The graph is translated to the left.
   d. The $x$-coordinate is $-2$.

6. a.
   b. $(3, 0); (4, 10); (2, -10)$
7.  a – d.

b. –3

c. 2

d. If \( y = \frac{1}{2}x \) is translated to the right 5 units, the new line passes through the point (5, 0).

8.  a. and b.

c. The \( y \)-coordinate is 5.

d. \( y = -3(4 - 4) + 5 = 5 \)

9.  a. and b.

c. \( y = \frac{1}{4}(x - 2) + 3 \)

d. The \( y \)-coordinate is 3.
10. a. 

b. The line \( y = 2x \) is translated to the left 1 and up 3.

c. \((-1, 3)\)

11. a. \( y = 4(x - 2) + 1 \)

b. \( y = 4(x + 2) - 1 \)

12. a. \( y = 4(x - 2) + 1 \)

b. \( y = 4(x + 2) - 1 \)

13. a. Since the slope and a point on the line are both visible in the equation, the form is called point-slope.

b. \( y = m(x - 0) + b \) The slope is \( m \) and the point is \((0, b)\).

14. a. The slope is 3, and a point is \((2, 4)\).

b. The slope is \( \frac{1}{2} \), and a point is \((-3, 5)\).

c. The slope is 2, and a point is \((-1, 6)\).

15. a. and b.

c. The \( y \)-intercept is not on the grid that is provided.

16. The line can be interpreted as a translation of the line \( y = mx \) to the right \( h \) units and up \( k \) units, or the line can be interpreted as a line with a slope of \( m \) that passes through the point \((h, k)\).
Translations of Linear Functions

1. a. Graph \( y = 2x \). Plot at least 5 points on the line.
   b. Translate the graph 4 units to the right by moving each point 4 units to the right.
   c. Write the equation of the translated line in slope-intercept form.
   
   d. Factor out the common factor in the translated equation.

   e. What are the coordinates of the point to which the point (0, 0) is translated?

   f. Where does the amount of the translation appear in the factored form of the equation from part (d)?

2. a. Graph \( y = 2x \). Plot at least 5 points on the line.
   b. Translate the graph 4 units to the left by moving each point 4 units to the left.
   c. Write the equation for the translated graph in slope-intercept form.

   d. Factor out the common factor in the translated equation.

   e. What are the coordinates of the point to which the point (0, 0) is translated?

   f. Where does the amount of the translation appear in the factored form of the equation from part (d)?
3. Examine the equations in part (d) of questions 1 and 2.
   a. How do the equations differ?

   b. How does this difference indicate the direction of the horizontal translation?

   c. Explain why \( y = 2(x + 4) \) and \( y = 2(x - (4)) \) represent the same line.

   d. What is the significance of the common factor?

4. a. By how much does \( y = 3(x - 2) \) translate the line \( y = 3x \) horizontally?

   b. In which direction, left or right, is \( y = 3x \) translated?

   c. Graph \( y = 3(x - 2) \) using a horizontal translation.

   d. What is the \( x \)-coordinate of the point on the line \( y = 3(x - 2) \) when the \( y \)-coordinate is 0?

5. a. By how much does \( y = 4(x + 2) \) translate the line \( y = 4x \) horizontally?

   b. In which horizontal direction is \( y = 4x \) translated?

   c. Graph \( y = 4(x + 2) \) using a horizontal translation.

   d. What is the \( x \)-coordinate of the point on the line \( y = 4(x + 2) \) with a \( y \)-coordinate of 0?
6. a. Graph \( y = 10(x - 3) \) using a horizontal translation.
   
   b. Use your graph to complete the missing values of the coordinates on the line:
      
      \((__, 0)\); \((4, ____)\); \((__, -10)\)

7. a. Graph \( y = \frac{1}{2}x \).

   b. Graph \( y = \frac{1}{2}(x + 3) \) using a horizontal translation.
      
      What is the \( x \)-coordinate of the point on this line with a \( y \)-coordinate of 0?

   c. Graph \( y = \frac{1}{2}(x - 2) \). What is the \( x \)-coordinate of the point on this line with a \( y \)-coordinate of 0?

   d. Plot the point \((5, 0)\) and draw a line with a slope of \( \frac{1}{2} \) through the point. The equation of this line will be \( y = \frac{1}{2}(x - 5) \).
      
      Explain how this is a horizontal translation of \( y = \frac{1}{2}x \).

8. a. Graph \( y = -3(x - 4) \) using a horizontal translation.
      
      Plot at least 5 points.

   b. Translate the graph in part (a) up 5 units.

   c. What is the \( y \)-coordinate of the point on the translated line when the \( x \)-coordinate is 4?

   d. The equation of the line can be written as \( y = -3(x - 4) + 5 \)
      
      Show that if \( x = 4 \) then \( y = 5 \).
9.  a. Graph \( y = \frac{1}{4}x \).
    b. Translate the line in part (a) to the right 2 units and up 3 units.
    c. Write the equation for the translated line in part (b) in the form, \( y = m(x - h) + k \), where \( h \) is the amount that the graph of \( y = \frac{1}{4}x \) is translated to the right and \( k \) is the amount that the line is translated up.
    d. What is the \( y \)-coordinate of the point on the translated line when the \( x \)-coordinate is 2?

10. a. Plot the point \((-1, 3)\) and graph a line through the point with a slope of 2.
    b. Explain how the line transforms \( y = 2x \).
    c. The point \((0, 0)\) on \( y = 2x \) is translated to what point on the line \( y = 2(x - (-1)) + 3 \)?

11. Write an equation in the form \( y = m(x - h) + k \) for the line that translates the line \( y = 4x \).
    a. 2 units to the right and up 1 unit.
    b. 2 units to the left and down 1 unit.

12. Using translations, write the equation of the line in the form \( y = m(x - h) + k \) that has a slope of 4 and:
    a. passes through the point \((2, 1)\).
    b. passes through the point \((-2, -1)\).
13. a. The form for the equation of a line \( y = m(x - h) + k \) is called the point-slope form. Justify this name.

b. The equation of a line written in the form \( y = mx + b \) is called the slope-intercept form of a line because both the slope and the \( y \)-intercept are visible in the equation. Show that the slope-intercept form for a line is the same as the point-slope form for a line with a slope of \( m \) that passes through the point \((0, b)\).

14. Without graphing the following lines, list the slope of the line and name a point on the line.
   a. \( y = 3(x - 2) + 4 \)
   b. \( y = \frac{1}{2}(x - (-3)) + 5 \)
   c. \( y = 2(x + 1) + 6 \)

15. a. Graph \( y = 2(x + 8) - 3 \).
   b. Graph \( y = 2x + 13 \).
   c. Explain why graphing this line on the provided grid is easier when the equation is given in point-slope form than in slope-intercept form.

16. Describe two ways in which the equation, \( y = m(x - h) + k \) where \( h > 0 \), can be graphed without converting it to slope-intercept form.